**Around the Poincaré lemma, after Beilinson [1]**

**Luc Illusie**

(Preliminary notes)

The following notes are an expanded version of talks given at the department of mathematics of the university of Padova on October 25 and 26, 2012, at the Séminaire RÉGA[1] on December 12, 2012, lectures at the KIAS (Seoul) on January 3 and 4, 2013, and a course at the Morningside Center of Mathematics (Beijing), on February 18, 25, and March 4, 11, 2013.

## 1. The classical Poincaré lemma and the Betti - de Rham comparison theorem

For a smooth analytic space $X/\mathbf{C}$, the classical Poincaré lemma[2] says that the natural augmentation

$$\mathbf{C}_X \to \Omega^{\cdot}_{X/\mathbf{C}} = (\mathcal{O}_X \to \Omega^1_{X/\mathbf{C}} \to \cdots) \tag{1.1}$$

is a quasi-isomorphism, where $\Omega^{\cdot}_{X/\mathbf{C}}$ is the de Rham complex of holomorphic differential forms on $X$. Actually, over any polydisc or more generally any star shaped open $U$ in $X$, (1.1) induces a homotopy equivalence $\mathbf{C} = \Gamma(U, \mathbf{C}_X) \to \Gamma(U, \Omega^{\cdot}_{X/\mathbf{C}})$, a homotopy operator being given by integration, $\omega \mapsto \int_0^1 i_{\partial_t} h^*(\omega)$, $h(t,x) = tx$.

From (1.1) one deduces

$$R\Gamma(X, \mathbf{C}) \xrightarrow{\sim} R\Gamma(X, \Omega^{\cdot}_{X/\mathbf{C}}), \tag{1.1.1}$$

$$H^*(X, \mathbf{C}) \xrightarrow{\sim} H^*(X, \Omega^{\cdot}_{X/\mathbf{C}}) \tag{1.1.2}$$

[1] Réseau Étudiants en Géométrie Algébrique http://www.math.u-psud.fr/~maculan/rega/index.php?p=2

[2] According to de Rham, ([7], p. 646) this lemma, attributed to Poincaré, was in fact first proved by Volterra. Here is what de Rham writes : "... dans ses Leçons sur la Géométrie des espaces de Riemann, dont la première édition a paru en 1928, E. Cartan donne le nom de Théorème de Poincaré au fait que la différentielle extérieure seconde d'une forme différentielle est toujours nulle, ce qui est d'ailleurs trivial et résulte immédiatement de la définition de cette différentielle extérieure. La réciproque, valable dans l'espace euclidien, n'est pas triviale, et dans ses Leçons sur les invariants intégraux, parues en 1922, E. Cartan démontre le théorème et sa réciproque sans mentionner Poincaré ni personne d'autre. Et aujourd'hui, c'est cette réciproque qui est assez couramment appelée lemme de Poincaré. Or ces propositions sont parfaitement énoncées et démontrées dans des travaux de Volterra (voir en particulier : Opere matematice, Vol. I, p. 407 et 422) datant de 1889 ; on y trouve aussi la formule de Stokes sous sa forme générale, ainsi d'ailleurs que - sous un autre nom il est vrai - la notion de forme harmonique dans l'espace euclidien. Il est clair que Cartan n'a pas eu connaissance de ces travaux, sinon il n'aurait pas manqué de les citer et de rendre justice à Volterra".

between Betti and de Rham cohomology (where we wrote $\mathbf{C}$ for $\mathbf{C}_X$).

Changing notation, suppose now that $X$ is a smooth *scheme* of finite type over $\mathbf{C}$, and let $X^{\mathrm{an}}$ be the associated analytic space. Let $\Omega^{\cdot}_{X/\mathbf{C}}$ be the de Rham complex of *algebraic* differential forms on $X$ (endowed with the Zariski topology). It is no longer true that the natural augmentation

$$\mathbf{C}_X \to \Omega^{\cdot}_{X/\mathbf{C}}$$

is a quasi-isomorphism. (The cohomology sheaves of $\Omega^{\cdot}_{X/\mathbf{C}}$ were studied by Bloch and Ogus [2], they have a deep connection with algebraic cycles on $X$.) It is of course not true either that the natural map $R\Gamma(X, \mathbf{C}) \to R\Gamma(X^{\mathrm{an}}, \mathbf{C})$ is an isomorphism ($H^i(X, \mathbf{C}) = 0$ for $i > 0$). However, by a theorem of Grothendieck [12], the natural map

$$R\Gamma(X, \Omega^{\cdot}_{X/\mathbf{C}}) \to R\Gamma(X^{\mathrm{an}}, \Omega^{\cdot}_{X^{\mathrm{an}}/\mathbf{C}}) \tag{1.2}$$

is an isomorphism. (For $X/\mathbf{C}$ proper it is a consequence of Serre's GAGA theorems. In the general case, it follows from the existence (by Hironaka) of good smooth compactifications, and a local calculation at infinity, a divisor with normal crossings.) Therefore, by composing with the inverse of (1.1.1) we get isomorphisms

$$R\Gamma(X, \Omega^{\cdot}_{X/\mathbf{C}}) \xrightarrow{\sim} R\Gamma(X^{\mathrm{an}}, \mathbf{C}), \tag{1.2.1}$$

$$H^*(X, \Omega^{\cdot}_{X/\mathbf{C}}) \xrightarrow{\sim} H^*(X^{\mathrm{an}}, \mathbf{C}). \tag{1.2.2}$$

One has a natural isomorphism

$$H^*(X^{\mathrm{an}}, \mathbf{C}) \xrightarrow{\sim} \mathrm{Hom}(H_*(X^{\mathrm{an}}), \mathbf{C}), \tag{1.2.3}$$

where $H_*(X^{\mathrm{an}})$ is the singular integral homology of $X^{\mathrm{an}}$. When $X$ is *affine*, $H^*(X, \Omega^{\cdot}_{X/\mathbf{C}})$ is just the cohomology of global sections of the de Rham complex,

$$H^*(X, \Omega^{\cdot}_{X/\mathbf{C}}) = H^*\Gamma(X, \Omega^{\cdot}_{X/\mathbf{C}})$$

and one can check (this is a form of de Rham's theorem) that the composite isomorphism

$$H^*\Gamma(X, \Omega^{\cdot}_{X/\mathbf{C}}) \xrightarrow{\sim} \mathrm{Hom}(H_*(X^{\mathrm{an}}), \mathbf{C}) \tag{1.2.4}$$

is given up to sign by integration of differential forms along singular simplices $\omega \mapsto (\gamma \mapsto \int_\gamma \omega)$ (according to ([6], 1.2) the sign is $(-1)^{n(n+1)/2}$ for $\omega$ of degree

$n$). The complex numbers $\int_\gamma \omega$ appearing in this way are called *periods*, and (1.2.2), (1.2.4) *period isomorphisms.*

*Example.* For $X = \mathbf{G}_m = \operatorname{Spec} \mathbf{C}[t, t^{-1}]$ the multiplicative group over $\mathbf{C}$, (1.2.4) takes $\omega = dt/t$ (whose class is a basis of $H^1(X, \Omega^\cdot_{X/\mathbf{C}})$) to the linear form sending the generator $x \mapsto e^{2\pi i x}$, $0 \leq x \leq 1$ of $H_1(X)$ to $-2\pi i$.

If $K$ is any field and $X$ is a $K$-scheme, one can consider its de Rham complex $\Omega^\cdot_{X/K}$ and its hypercohomology, $H^*(X, \Omega^\cdot_{X/K})$. When $X/K$ is smooth of finite type and $K$ is a subfield of $\mathbf{C}$ via some embedding $\sigma : K \hookrightarrow \mathbf{C}$, and $X_\sigma$ is the pull-back of $X$ to $\mathbf{C}$ via $\sigma$, one has a canonical isomorphism

$$H^*(X, \Omega^\cdot_{X/K}) \otimes_{K,\sigma} \mathbf{C} \xrightarrow{\sim} H^*(X_\sigma, \Omega^\cdot_{X_\sigma/\mathbf{C}}),$$

hence, by (1.2.2), a period isomorphism

$$H^*(X, \Omega^\cdot_{X/K}) \otimes_{K,\sigma} \mathbf{C} \xrightarrow{\sim} H^*(X_\sigma^{\mathrm{an}}, \mathbf{C}) = H^*(X_\sigma^{\mathrm{an}}, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C} \tag{1.2.5}$$

(depending on $\sigma$). Thus, for $K = \mathbf{Q}$ we get *two rational* $\mathbf{Q}$*-structures* on $H^*(X^{\mathrm{an}}, \mathbf{C})$, related by the period isomorphism (1.2.5) which is highly non rational, as the example above shows.

When $X/K$ is proper and smooth, (1.2.5) is more than an equality of dimensions. Both sides have a natural filtration (the *Hodge filtration*), and (1.2.5) is compatible with it. One can extend this isomorphism to the case $X/K$ is only separated and of finite type, by replacing the de Rham cohomology on the left hand side by one defined by suitable hypercoverings like in Deligne Hodge III [5]. We'll come back to this later.

## 2. The $p$-adic étale - de Rham comparison theorem : outline

Let $K$ be a field of characteristic zero, $\overline{K}$ an algebraic closure of $K$, with Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$. Let $X/K$ be separated and of finite type. For any prime $\ell$, $H^*(X_{\overline{K}}, \mathbf{Q}_\ell)$ is a finite dimensional $\mathbf{Q}_\ell$-representation of $G_K$. If we have an embedding $\sigma : K \hookrightarrow \mathbf{C}$ and choose an extension $\overline{\sigma} : \overline{K} \hookrightarrow \mathbf{C}$, we have a comparison isomorphism (depending on $\overline{\sigma}$ in a $G_K$-equivariant way) :

$$H^*(X_{\overline{K}}, \mathbf{Q}_\ell) \xrightarrow{\sim} H^*(X_\sigma^{\mathrm{an}}, \mathbf{Q}) \otimes \mathbf{Q}_\ell.$$

But in general, for $X/K$ proper and smooth, we have no way to compare this representation of $G_K$ with the de Rham cohomology $H^*(X, \Omega^\cdot_{X/K})$. The situation, however, is better when $K$ is a *p-adic field* and $\ell = p$. In this case, we have a comparison isomorphism involving Fontaine's ring $B_{\mathrm{dR}}$, which, after partial results by Bloch-Kato, Fontaine-Messing, Hyodo-Kato, was proved in full generality in the last 15 years by several authors (and different methods) (Tsuji [20], Faltings [8], Niziol [17], Yamashita [21]), and quite recently

Beilinson [1]. I will explain Beilinson's approach, which is close in spirit to the construction of (1.2.2) via the Poincaré lemma (1.1).

Fix a complete discrete valuation field $K$ of characteristic 0, with perfect residue field $k$ of characteristic $p > 0$, ring of integers $\mathcal{O}_K$, and as above an algebraic closure $\overline{K}$ of $K$, with Galois group $G_K$. Let $X/K$ be separated and of finite type. We will define a comparison map

$$\rho_{\mathrm{dR}} : R\Gamma_{\mathrm{dR}}(X/K) \otimes B^+_{\mathrm{dR}} \to R\Gamma(X_{\overline{K}}, \mathbf{Z}_p) \otimes B^+_{\mathrm{dR}}, \tag{2.1}$$

compatible with all structures (Galois actions, Hodge filtrations on $H^*$), whose extension to $B_{\mathrm{dR}}$,

$$\rho_{\mathrm{dR}} : R\Gamma_{\mathrm{dR}}(X/K) \otimes B_{\mathrm{dR}} \to R\Gamma(X_{\overline{K}}, \mathbf{Q}_p) \otimes B_{\mathrm{dR}} \tag{2.2}$$

is an isomorphism.

Here $B^+_{\mathrm{dR}}$, $B_{\mathrm{dR}}$ are Fontaine rings (whose definition will be recalled later), and $R\Gamma_{\mathrm{dR}}(Y/\kappa)$ for any $Y$ separated and of finite type over a field $\kappa$ of char. 0 is defined in the following way:

(a) for $Y/\kappa$ proper and smooth,

$$R\Gamma_{\mathrm{dR}}(Y/\kappa) := R\Gamma(Y, \Omega^{\cdot}_{Y/k}),$$

(b) for $Y/\kappa$ smooth (but not necessarily proper),

$$R\Gamma_{\mathrm{dR}}(Y/\kappa) := R\Gamma(\overline{Y}, \Omega^{\cdot}_{\overline{Y}/\kappa}(\log D))$$

for any proper and smooth compactification $\overline{Y}$ of $Y$ such that $D := \overline{Y} - Y$ is a divisor with normal crossings (by Deligne's Hodge II [4], the right hand side does not depend on the choice of $\overline{Y}$). In both cases, $H^*_{\mathrm{dR}}(Y/\kappa)$ is endowed with the *Hodge filtration* defined by the naive filtration of the de Rham complexes, which filtration does not depend on the choices ;

(c) in general, choose a proper hypercovering $V_{\cdot} \to Y$, with each $V_n/\kappa$ smooth, and a simplicial compactification $V_{\cdot} \hookrightarrow \overline{V}_{\cdot}$ such that each $\overline{V_{n\cdot}}/\kappa$ is proper and smooth, and the complement $\overline{V}_n - V_n$ is a divisor with normal crossings $D_n$, and define

$$R\Gamma_{\mathrm{dR}}(Y/\kappa) := R\Gamma(\overline{V}_{\cdot}, \Omega^{\cdot}_{\overline{V}_{\cdot}/\kappa}(\log D_{\cdot})).$$

By Deligne's Hodge III [5], the right hand side does not depend on the choices, and the Hodge filtration on $H^*_{\mathrm{dR}}(Y/\kappa)$, defined by the naive filtration of $\Omega^{\cdot}_{\overline{V}_{\cdot}/\kappa}(\log D_{\cdot})$ is independent of the choices.

*Strategy of the construction.*

Beilinson's method to construct (2.1) uses *no syntomic cohomology, no algebraic $K$-theory, no almost étale theory.* It is not a by-product of a finer comparison theorem involving Frobenius structures ($C_{\mathrm{st}}$, or $C_{\mathrm{pst}}$). Instead, it relies on a $p$-adic variant of the Poincaré lemma (1.1), based on de Jong's alterations and a $p$-divisibility theorem generalizing a vanishing theorem of Bhatt, with, as a technical tool, the theory of derived de Rham complexes.

In the construction of (1.2.1) the bridge between the two sides is given by the de Rham complex $\Omega^{\cdot}_{X^{\mathrm{an}}/\mathbf{C}}$ on $X^{\mathrm{an}}$, which, on the one hand, is a resolution of the constant sheaf $\mathbf{C}$, and on the other, receives (or more precisely is the analytification of) the algebraic de Rham complex $\Omega^{\cdot}_{X/\mathbf{C}}$ :

$$\mathbf{C} \to \Omega^{\cdot}_{X^{\mathrm{an}}/\mathbf{C}} \leftarrow \Omega^{\cdot}_{X/\mathbf{C}}. \tag{2.3}$$

For any field $F$, denote by $\mathcal{V}ar_F$ the category of *$F$-varieties*, i. e. reduced, separated $F$-schemes of finite type. The bridge between the two sides of (2.1) is given by a certain projective system of sheaves of filtered differential graded algebras (dga for short)

$$\mathcal{A}^{\natural}_{\mathrm{dR}}$$

on $\mathcal{V}ar_{\overline{K}}$, equipped with *Voevodsky's $h$-topology*, which is generated by proper surjective maps and surjective étale coverings. The comparison pattern is basically of the form (2.3) :

$$\begin{tikzcd} \mathcal{A}^{\natural}_{\mathrm{dR}}\widehat{\otimes}\mathbf{Z}_p & \mathcal{A}^{\natural}_{\mathrm{dR}} \arrow[l] \arrow[r] & \mathcal{A}^{\natural}_{\mathrm{dR}} \otimes \mathbf{Q} \arrow[d, "a"] \\ A_{\mathrm{dR}}\widehat{\otimes}\mathbf{Z}_p \arrow[u, "b"] & & \mathcal{A}_{\mathrm{dR}} \end{tikzcd} \tag{2.4}$$

A few explanations on the items appearing in (2.4) :

• $\mathcal{A}_{\mathrm{dR}}$ is a projective system of sheaves of filtered $\overline{K}$-dga $\mathcal{A}_{\mathrm{dR}}/F^i$ on $\mathcal{V}ar_{\overline{K}}$ (equipped with the $h$-topology), which has the property that for any $\overline{K}$-variety $Y$, we have a natural identification

$$R\varprojlim R\Gamma(Y, \mathcal{A}_{\mathrm{dR}}/F^i) \simeq R\Gamma_{\mathrm{dR}}(Y/\overline{K}), \tag{2.5}$$

the isomorphism induced on $H^*$ being compatible with the filtrations on both sides. The construction of $\mathcal{A}_{\mathrm{dR}}$ (as that of its refined variant $\mathcal{A}^{\natural}_{\mathrm{dR}}$) is algebraic, but the above identification uses mixed Hodge theory. For $Y/\overline{K}$ proper and smooth, the underlying dga of $\mathcal{A}_{\mathrm{dR}}$ is just $\Omega^{\cdot}_{Y/\overline{K}}$.

• The symbol $\widehat{\otimes}$ denotes a completed derived tensor product : for any complex $E$ of abelian groups (or sheaves of abelian groups) Beilinson defines

$$E\widehat{\otimes}\mathbf{Z}_p = R\varprojlim_n (E \otimes^L \mathbf{Z}/p^n).$$

• Recall the definition of Fontaine's ring

$$B_{\mathrm{dR}}^{+} := \varprojlim_{i}(((\mathcal{O}_K \otimes_W W(\varprojlim_{x\mapsto x^p} \mathcal{O}_{\overline{K}}/p))/J^i)\hat{}\otimes \mathbf{Q})$$

where $W = W(k)$, $\theta : \mathcal{O}_K \otimes_W W(\varprojlim_{x\mapsto x^p} \mathcal{O}_{\overline{K}}/p) \to \mathcal{O}_C$ is the canonical map, sending

$$(x_0, x_1, \cdots, x_n, \cdots) \in W(\varprojlim_{x\mapsto x^p} \mathcal{O}_{\overline{K}}/p)$$

to $\sum p^n x_n^{(n)}$ (where $x_n = (x_n^{(m)})_m$ with $x_n^{(m)} \in \mathcal{O}_{\overline{K}}$, and $(x_n^{(m+1)})^p = x_n^{(m)}$), and $C := \widehat{\overline{K}}$, $J = \operatorname{Ker}\theta$, and $\hat{}$ means $p$-adic completion.

$A_{\mathrm{dR}}$ is a projective system of filtered $\mathcal{O}_K$-dga $A_{\mathrm{dR}}/F^i$, which is related to Fontaine's ring by a canonical isomorphism (compatible with the induced filtrations)

$$B_{\mathrm{dR}}^{+} \xrightarrow{\sim} R\varprojlim_{i}(((A_{\mathrm{dR}}/F^i)\widehat{\otimes}\mathbf{Z}_p) \otimes \mathbf{Q}). \tag{2.6}$$

• The horizontal maps are the obvious ones.

• $a$ and $b$ are isomorphisms. The *p-adic Poincaré lemma* is the fact that $b$ is an isomorphism, more precisely that, for any $n \geq 1$, and any $i$, the map

$$(A_{\mathrm{dR}}/F^i) \otimes^L \mathbf{Z}/p^n \to (\mathcal{A}_{\mathrm{dR}}^{\natural}/F^i) \otimes^L \mathbf{Z}/p^n \tag{2.7}$$

is an isomorphism. The fact that $a$ is an isomorphism is formal.

## 3. Preliminaries to the construction of $\mathcal{A}_{\mathrm{dR}}^{\natural}$

There are two basic ingredients : (1) derived de Rham complexes, (2) de Jong's semi-stable models.

(1) *Derived de Rham complexes.*

(a) *The case of schemes*

Let's start with the affine case. Let $A$ be a (commutative) ring. The forgetful functor $U$ : $A$-alg $\to$ Sets from the category of $A$-algebras to the category of sets admits a left adjoint $T$ : Sets $\to$ $A$-alg, associating to a set $I$ the free $A$-algebra $T(I) := A[I] = S_A(A^{(I)})$ on the set $I$. By a well-known construction (cf. e. g. ([13], I 1.5)), for any $A$-algebra $B$ this pair of adjoint functors $(T, U)$ gives rise to a simplicial $A$-algebra $P_A(B)$ augmented to $B$,

$$P_A(B) = (\cdots A[A[B]] \rightrightarrows A[B]) \to B, \tag{3.1}$$

having the following properties :

(i) in each degree $n$, $P_A(B)_n$ is a free $A$-algebra (namely, $P_A[B]_n = A[P_A(B)_{n-1}]$, with the convention that $P_A(B)_{-1} = B$),

(ii) the augmentation induces a quasi-isomorphism on the underlying chain complexes.

(More generally, an augmented simplicial $A$-algebra $R \to B$ having properties (i) and (ii) is called a *free resolution* of $B$. For $R = P_A[B]$, the augmentation actually induces a homotopy equivalence on the underlying simplicial sets.)

Let $P := P_A(B)$. Applying the functor of Kähler differentials $\Omega^1_{-/-}$ to (3.1) we get a simplicial $P$-module $\Omega^1_{P/A}$, which by extension of scalars to $B$ is turned into a simplicial $B$-module, whose underlying chain complex is by definition the *cotangent complex of* $B$ *over* $A$, denoted

$$L_{B/A} := B \otimes_P \Omega^1_{P/A}. \tag{3.2}$$

It depends functorially (as a complex) on $B/A$. One has $H^i(L_{B/A}) = 0$ for $i > 0$, and $H^0(L_{B/A}) = \Omega^1_{B/A}$. One shows that, up to an isomorphism in $D(B)$, $L_{B/A}$ can be calculated by replacing $P$ by any free resolution of $B$. If $A$ is noetherian and $B$ of finite type over $A$, such resolutions exist which are finitely generated in each degree. In this case, $H^i(L_{B/A})$ is finitely generated over $B$ for all $i$.

Let now $f : X \to Y$ be a morphism of schemes. A similar construction yields the *cotangent complex*

$$L_{X/Y}.$$

Namely, consider the morphism of sheaves of algebras $A = f^{-1}(\mathcal{O}_Y) \to B = \mathcal{O}_X$ on the Zariski site of $X$. For any sheaf of sets $E$ on $X$, let $A[E]$ be the (commutative) free $A$-algebra on $E$ $(= S_A(A^{(E)}))$. The functor $T : E \mapsto A[E]$ from the category $Sh_X$ of sheaves of sets on $X$ to the category $A$-alg of sheaves of $A$-algebras on $X$ is left adjoint to the forgetful functor $U$ associating with an $A$-algebra $B$ the underlying sheaf of sets. This pair of adjoint functors $(T, U)$ between $Sh_X$ and $A$-alg again gives rise to a simplicial $A$-algebra $P_A(B)$ augmented to $B$,

$$P_A(B) = (\cdots A[A[B]] \rightrightarrows A[B]) \to B, \tag{3.3}$$

having the following properties :

(i) in each degree $n$, $P_A(B)_n$ is a free $A$-algebra on a sheaf of sets

(ii) the augmentation induces a quasi-isomorphism on the underlying chain complexes.

(As above, an augmented simplicial $A$-algebra $R \to B$ having properties (i) and (ii) is called a *free resolution* of $B$, and for $R = P_A[B]$, the augmentation induces a homotopy equivalence on the underlying simplicial sheaves.)

Again, putting $P := P_A(B)$ and applying the functor of Kähler differentials $\Omega^1_{-/-}$ to (3.3) we get a simplicial $P$-module $\Omega^1_{P/A}$, which by extension of scalars to $B$ is turned into a simplicial $B$-module, whose underlying chain complex is by definition the cotangent complex of $X$ over $Y$, denoted

$$L_{X/Y} := B \otimes_P \Omega^1_{P/A}. \tag{3.4}$$

It depends functorially (as a complex) on $X/Y$. Its cohomology sheaves $\mathcal{H}^i L_{X/Y}$ are quasi-coherent, zero for $i > 0$, and $\mathcal{H}^0 L_{X/Y} = \Omega^1_{X/Y}$. Again, one shows that, up to an isomorphism in $D(B)$, $L_{X/Y}$ can be calculated by replacing $P$ by any free resolution of $B$. Moreover, If $X = \operatorname{Spec} S \to Y = \operatorname{Spec} R$ is a map of affine schemes, one has a natural quasi-isomorphism ([13], II 2.3.6.3)

$$(L_{S/R})\tilde{} \xrightarrow{\sim} L_{X/Y}, \tag{3.5}$$

where $(-)\tilde{}$ denotes an associated quasi-coherent sheaf. Any composition $X \xrightarrow{f} Y \to S$ gives rise to an exact *transitivity triangle* ([13], II 2.1)

$$f^* L_{Y/S} \to L_{X/S} \to L_{X/Y} \to . \tag{3.6}$$

If $i : X \to Z$ is an embedding of $X$ into a smooth scheme $Z/S$, with ideal $I$, then we have a canonical isomorphism in $D(X)$ ([13], III 1.2)

$$\tau_{\geq -1} L_{X/S} \xrightarrow{\sim} [I/I^2 \xrightarrow{d_{Z/S}} i^* \Omega^1_{Z/S}], \tag{3.7}$$

where $\tau_{\geq i}$ is the canonical truncation functor. If moreover $i$ is a *regular immersion*, then ([13] III 3.2), in $D(X)$,

$$L_{X/S} \xrightarrow{\sim} \tau_{\geq -1} L_{X/S} \xrightarrow{\sim} [I/I^2 \xrightarrow{d_{Z/S}} i^* \Omega^1_{Z/S}]. \tag{3.8}$$

*Example* 3.9. Applying (3.7) to $\operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$ where $L$ runs through the finite extensions of $K$ contained in $\overline{K}$, one finds that

$$L_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \xrightarrow{\sim} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}. \tag{3.9.1}$$

Recall that by Fontaine ([10], th. 1)

$$\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} = (\overline{K}/\mathfrak{a})(1) \tag{3.9.2}$$

for the fractional ideal $\mathfrak{a}$ of $\mathcal{O}_{\overline{K}}$ generated by $p^{-1/(p-1)}\mathfrak{D}^{-1}_{K/K_0}$, where $K_0$ is the fraction field of $W(k)$ and $\mathfrak{D}_{K/K_0}$ the different :

$$\mathfrak{a} := p^{-1/(p-1)}\mathfrak{D}^{-1}_{K/K_0}.\mathcal{O}_{\overline{K}} \subset \overline{K}.$$

We'll return to this later.

Coming back to the general situation $f : X \to Y$, with $P = P_A(B)$, the de Rham complex

$$\Omega^{\cdot}_{P/A} = (P \to \Omega^1_{P/A} \to \cdots \to \Omega^n_{P/A} \to \cdots)$$

is a simplicial differential graded $A$-algebra, which we can turn into a differential graded algebra, by taking the corresponding bicomplex (of $A$-modules) and the associated total complex

$$L\Omega^{\cdot}_{X/Y} := \mathbf{s}\Omega^{\cdot}_{P/A}, \tag{3.10}$$

$$(\mathbf{s}\Omega^{\cdot}_{P/A})^n = \oplus_{i+j=n}\Omega^j_{P_{-i}/A}.$$

We call $L\Omega^{\cdot}_{X/Y}$ the *derived de Rham complex* of $X/Y$ ([14], VIII 2.1) (see also Errsln283.pdf on Illusie's web page). The multiplicative structure is given by the product on the exterior algebra $\Omega^*_{P/A}$ and the product on the chain complex of $P$ defined by the shuffle map $P_i \otimes P_j \to P_{i+j}$ (sending $x \otimes y$ to $\sum \varepsilon(\mu, \nu) s_{\nu_1} \cdots s_{\nu_j} x.s_{\mu_1} \cdots s_{\mu_i} y$, where $(\mu, \nu)$ runs through the $(i, j)$-shuffles of $(1, \cdots, i + j)$ and $\varepsilon(\mu, \nu)$ is the signature of $(\mu, \nu)$). It has a natural filtration (compatible with the dga structure)

$$F^i L\Omega^{\cdot}_{X/Y} := \mathbf{s}\Omega^{\geq i}_{P/A}. \tag{3.11}$$

We view $L\Omega^{\cdot}_{X/Y}$ as an object of the derived category of filtered $A$-dga on $X$ (one can show as such an object it can be calculated by replacing the canonical resolution $P$ by any free resolution of the $A$-algebra $B$). By the equivalence between the derived categories of simplicial modules over $P$ and $B$ (cf. [13], I 3.3.2.1) the associated graded can be viewed in a natural way as an object of $D(X)$ (unbounded in general). We have

$$\mathrm{gr}^1_F L\Omega^{\cdot}_{X/Y} = L_{X/Y}[-1] \tag{3.12}$$

and

$$\mathrm{gr}^{\cdot}_F L\Omega^{\cdot}_{X/Y} = (L\Lambda^{\cdot} L_{X/Y})[-.]. \tag{3.13}$$

For the $p$-adic de Rham comparison theorem, the object of interest is the *pro-completion* of $L\Omega^{\cdot}_{X/Y}$, i.e. the projective system

$$L\widehat{\Omega}^{\cdot}_{X/Y} := \text{“}\varprojlim_i\text{”} L\Omega^{\cdot}_{X/Y}/F^i, \tag{3.14}$$

viewed as a (pro-) filtered dga on $X$. For $X = \operatorname{Spec} S \to Y = \operatorname{Spec} R$ as above we define similarly $L\Omega^{\cdot}_{S/R}$, $F^i$, etc., the complexes of sheaves associated to $\mathrm{gr}_F L\Omega^{\cdot}_{S/R}$ are naturally quasi-isomorphic to $\mathrm{gr}_F L\Omega^{\cdot}_{X/Y}$.

*Example* 3.9 (cont'd) : *a new look at Fontaine's ring* $B^+_{\mathrm{dR}}$.

Beilinson defines

$$A_{\mathrm{dR}} := L\widehat{\Omega}^{\cdot}_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}. \tag{3.9.3}$$

Let us sketch the proof of the isomorphism (2.6). Recall the canonical isomorphisms

$$L_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \xrightarrow{\sim} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \xrightarrow{\sim} (\overline{K}/\mathfrak{a})(1) = (\mathbf{Q}_p/\mathbf{Z}_p) \otimes \mathfrak{a}(1).$$

We have

$$(\mathbf{Q}_p/\mathbf{Z}_p)\widehat{\otimes}\mathbf{Z}_p = \mathbf{Z}_p[1],$$

and, by Quillen's shift formula (cf. ([13], I 4.3.2.1)), for a module $M$ over a ring $A$ in some topos,

$$L\Lambda^i(M[1]) \xrightarrow{\sim} L\Gamma^i(M)[i]$$

for any $i \geq 0$. It follows that

$$H^n(\mathrm{gr}_F A_{\mathrm{dR}}\widehat{\otimes}\mathbf{Z}_p) = \begin{cases} 0 & \text{if } n \neq 0, \\ \mathcal{O}_C\langle\widehat{\mathfrak{a}}(1)\rangle & \text{if } n = 0, \end{cases} \tag{3.9.4}$$

where $\mathcal{O}_C = \widehat{\mathcal{O}_{\overline{K}}}$ and $\mathcal{O}_C\langle\rangle$ denotes a divided power algebra, and hence that $(A_{\mathrm{dR}}/F^{i+1})\widehat{\otimes}\mathbf{Z}_p$ is concentrated in degree zero, with

$$((A_{\mathrm{dR}}/F^{i+1})\widehat{\otimes}\mathbf{Z}_p) \otimes \mathbf{Q} \simeq C[t]/t^{i+1}.$$

Now observe that

$$A_{\mathrm{dR}}/F^2 = (\mathcal{O}_{\overline{K}} \xrightarrow{d} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$$

hence

$$A_{\mathrm{dR}}/F^2 \xrightarrow{\sim} \mathcal{O}'_{\overline{K}} := \operatorname{Ker} d$$

as $d$ is surjective (by Fontaine ([11],1.4.4)). If

$$u_i : A_{\mathrm{inf}}/F^{i+1} \to (A_{\mathrm{dR}}/F^{i+1})\widehat{\otimes}\mathbf{Z}_p$$

is the canonical map, where

$$A_{\mathrm{inf}} = \varprojlim_i((\mathcal{O}_K \otimes_W W(\varprojlim_{x\mapsto x^p} \mathcal{O}_{\overline{K}}/p))/J^i)^{\frown}$$

(the hat meaning $p$-adic completion) is the universal thickening, the map

$$u_1 : A_{\rm inf}/F^2 \to (A_{\rm dR}/F^2)\widehat{\otimes}\mathbf{Z}_p$$

is a filtered isomorphism by ([11], 1.4.3), hence also

$$u_{i\mathbf{Q}} : B^+_{\rm dR}/F^{i+1} \to (A_{\rm dR}/F^{i+1})\widehat{\otimes}\mathbf{Q}_p := ((A_{\rm dR}/F^{i+1})\widehat{\otimes}\mathbf{Z}_p) \otimes \mathbf{Q}.$$

Finally, taking projective limits, one gets the filtered isomorphism (2.6).

The derived completion $\widehat{\otimes}$ is essential in this calculation. For the non-completed graded we have by (3.13)

$$\mathrm{gr}^i_F A_{\rm dR} = L\Lambda^i\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}[-i].$$

Replacing $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ by the complex of flat $\mathcal{O}_{\overline{K}}$-modules $[\mathfrak{a} \to \overline{K}](1)$, concentrated in degrees -1 and 0, and using that $L\Lambda^i$ of a complex $[L \to M]$ of flat modules (in degrees -1 and 0) can be calculated by the Koszul complex

$$[\Gamma^i L \to \Gamma^{i-1}L \otimes M \to \cdots \to L \otimes \Lambda^{i-1}M \to \Lambda^i M]$$

(in degrees in $[-i, 0]$) (a formula similar to that of ([14], VIII 2.1.2.1)), we find that, for $i > 0$,

$$L\Lambda^i\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} = [\Gamma^i\mathfrak{a} \to \overline{K}](i)[i-1]$$

hence

$$\mathrm{gr}^i_F A_{\rm dR} = \begin{cases} \mathcal{O}_{\overline{K}} & \text{if } i = 0 \\ (\mathbf{Q}_p/\mathbf{Z}_p) \otimes \frac{1}{i!}\widehat{\mathfrak{a}}^i(i)[-1] & \text{if } i > 0. \end{cases} \tag{3.9.5}$$

It follows that (as a pro-object)

$$A_{\rm dR} \otimes \mathbf{Q} = \overline{K},$$

and the map $A_{\rm dR} \otimes \mathbf{Q} \to A_{\rm dR}\widehat{\otimes}\mathbf{Q}_p := (A_{\rm dR}\widehat{\otimes}\mathbf{Z}_p) \otimes \mathbf{Q}$ corresponds to the inclusion $\overline{K} \hookrightarrow B^+_{\rm dR}$.

As Beilinson observes, one can also deduce (3.9.5) from (3.9.4) : for $i > 0$, $\mathrm{gr}^i_F A_{\rm dR} \otimes^L \mathbf{Z}/p = (\mathrm{gr}^i_F A_{\rm dR}\widehat{\otimes}\mathbf{Z}_p) \otimes^L \mathbf{Z}/p = \Gamma^i\mathfrak{a}(i) \otimes \mathbf{Z}/p$, hence $\mathrm{gr}^i_F A_{\rm dR}$ is concentrated in degree 1, and given by (3.9.5).

(b) *Logarithmic variants.*

As de Jong's alterations produce compactifications with normal crossings at infinity and semi-stable pairs, hence differential forms with log poles, we will need variants of the above constructions for morphisms of *log schemes.*

Recall that a log scheme $(X, M)$ is a triple $(X, M, \alpha : M \to \mathcal{O}_X)$, where $M$ is a sheaf of monoids on $X$ *for the étale topology*, and $\alpha$ a homomorphism (for the multiplicative structure of $\mathcal{O}_X$), such that $\alpha$ induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) \xrightarrow{\sim} \mathcal{O}_X^*$. Morphisms of log schemes are defined in the obvious way. We refer to [16] for the basic language of log schemes.

If $f : (X, M) \to (Y, L)$ is a morphism of log schemes, a variant, due to Gabber and Olsson (see [18]) of the construction of the free resolution $P_A(B)$, applied to the morphism of *log rings* $(f^{-1}L \to f^{-1}\mathcal{O}_Y) \to (M \to \mathcal{O}_X)$, and of the definition of $L_{X/Y}$ produces a complex

$$L_{(X,M)/(Y,L)}, \tag{3.15}$$

concentrated in degree $\leq 0$, called the *cotangent complex* of $f$ (see §7 for a quick review of this construction). It again depends functorially on $f$ as a complex. We have $\mathcal{H}^0 L_{(X,M)/(Y,L)} = \Omega^1_{(X,M)/(Y,L)}$, the sheaf of Kähler (log) differential 1-forms, and if $(X, M)$ and $(Y, L)$ are *fine*, then the sheaves $\mathcal{H}^i L_{(X,M)/(Y,L)}$ are all quasi-coherent. If the log structures $L$ and $M$ are trivial, or more generally if $M$ is induced by $L$, then the natural map $L_{X/Y} \to L_{(X,M)/(Y,L)}$ is an isomorphism, where $L_{X/Y}$ is the cotangent complex of the underlying morphism of schemes. Any composition $(X, M) \xrightarrow{f} (Y, L) \to (S, K)$ produces an exact transitivity triangle

$$f^* L_{(Y,L)/(S,K)} \to L_{(X,M)/(S,K)} \to L_{(X,M)/(Y,L)} \to \tag{3.16}$$

similar to (3.5).

If $(X, M) \to (Y, L)$ is *log smooth* and, in addition, is *integral* (a technical condition[3] satisfied for example in the case of semi-stable reduction or if the base is a field with trivial log structure), then

$$L_{(X,M)/(Y,L)} \xrightarrow{\sim} \Omega^1_{(X,M)/(Y,L)}, \tag{3.17}$$

a locally free module of finite rank.

We have a *derived log de Rham complex*

$$(L\Omega^{\cdot}_{(X,M)/(Y,L)}, F^i), \tag{3.18}$$

which is a filtered $f^{-1}\mathcal{O}_Y$-dga defined similarly to (3.10), (3.11), with

$$\mathrm{gr}^1 L\Omega^{\cdot}_{(X,M)/(Y,L)} \simeq L_{(X,M)/(Y,L)}[-1], \tag{3.19}$$

[3] This means that for any geometric point $x$ of $X$ with image $y$ in $Y$, $\mathbf{Z}[M_x/\mathcal{O}^*_{(X,x)}]$ is flat over $\mathbf{Z}[L_y/\mathcal{O}^*_{(Y,y)}]$, see ([16], 4.1).

$$\text{(3.20)} \qquad \mathrm{gr} L\Omega^{\cdot}_{(X,M)/(Y,L)} \simeq L\Lambda(L_{(X,M)/(Y,L)})[-.].$$

As in (3.14) we define the *pro-completion* of $L_{(X,M)/(Y,L)}$, i. e. the projective system

$$\text{(3.21)} \qquad L\widehat{\Omega}^{\cdot}_{(X,M)/(Y,L)} := \text{“}\varprojlim_i\text{”} L\Omega^{\cdot}_{(X,M)/(Y,L)}/F^i.$$

(2) *De Jong's semi-stable models.*

Recall de Jong's theorem : if $X$ is a $K$-variety, there exists a finite extension $K'$ of $K$, an alteration $V \to X_{K'}$ (i. e. a proper, surjective map, sending each generic point to a generic point with a finite residue extension), and a compactification $V \subset \overline{V}$, with $\overline{V}$ projective over $\mathcal{O}_{K'}$, such that the pair $(V, \overline{V} - V)$ is a semi-stable pair over $\mathcal{O}_{K'}$, which means that $\overline{V}$ is regular, $\overline{V} - V$ is a reduced divisor with normal crossings (union of the special fiber $\overline{V}_{k'}$ over the residue field $k'$ of $\mathcal{O}_{K'}$ and a horizontal divisor $D$). Locally around a point of $\overline{V}_{k'}$, $\overline{V}$ is étale over $\mathcal{O}_{K'}[t_1, \cdots, t_n]/(t_1 \cdots t_r - \pi')$ (where $\pi'$ is a uniformizing parameter of $\mathcal{O}_{K'}$), and $D$ is given by the vanishing of $t_{r+1} \cdots t_s$.

Actually, we will need the following stronger form (also due to de Jong) : if $X$ as above is compactified into $\overline{X}$ proper over $\mathcal{O}_K$ (such a compactification exists by Nagata), then one can find a pair $(V, \overline{V})$ as above such that the alteration $V \to X_{K'}$ is the restriction to $X_{K'}$ of an alteration $\overline{V} \to \overline{X}_{\mathcal{O}_{K'}}$.

In order to deal with the extensions needed to get semi-stable reduction, it is convenient to adopt Beilinson's conventions and terminology, which are slightly different from those of de Jong : a *semi-stable pair over $K$* is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d] & \overline{U} \arrow[d, "f"] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} \mathcal{O}_K \end{tikzcd} \quad ,$$

where $\overline{U}$ is regular, $j$ is a dense open immersion, $\overline{U} - U$ is a divisor with normal crossings, $f$ is proper and flat, and, if $g = \overline{U} \to \operatorname{Spec} \mathcal{O}_{K_U}$ is the Stein factorization of $f$, the closed fibers of $g$ are reduced. Here $\mathcal{O}_{K_U} := \Gamma(\overline{U}, \mathcal{O})$, and $K_U := \Gamma(\overline{U}_K, \mathcal{O})$.

One defines in the obvious way the notion of *semi-stable pair* over $\overline{K}$ (a pair $(U, \overline{U})$ of $\overline{K}$-varieties whose connected components come by base change via some point $\operatorname{Spec} \overline{K} \to \operatorname{Spec} \mathcal{O}_{K_V}$ from a semi-stable pair $(V, \overline{V})$ over $K$). They form a category denoted $\mathcal{V}ar^{\mathrm{ss}}_{\overline{K}}$. We have a forgetful functor

$$\phi : \mathcal{V}ar^{\mathrm{ss}}_{\overline{K}} \to \mathcal{V}ar_{\overline{K}},$$

$(U,\overline{U}) \mapsto U$, which is *faithful*, as our $\overline{K}$-varieties are reduced. Endow $\mathcal{V}ar_{\overline{K}}$ with the Voevodsky h-topology. Using de Jong's theorem (in its strong form), one checks that $\phi$ enjoys the following property :

(B) For any $V \in \mathcal{V}ar_{\overline{K}}$ and any finite family $((U_\alpha, \overline{U}_\alpha) \in \mathcal{V}ar^{\mathrm{ss}}_{\overline{K}}, f_\alpha : V \to U_\alpha)$, there exists an h-covering family $(U'_\beta \to V)$, with $U'_\beta = \phi(U'_\beta, \overline{U}'_\beta)$, such that any composition $U'_\beta \to V \to U_\alpha$ is induced by a (necessarily unique) map of pairs $(U'_\beta, \overline{U}'_\beta) \to (U_\alpha, \overline{U}_\alpha)$.

Property (B) implies that $\mathcal{B} := \mathcal{V}ar^{ss}_{\overline{K}}$, via $\phi$, plays the role of a *base* for the h-topology of $\mathcal{V}ar_{\overline{K}}$. More precisely, if one endows $\mathcal{V}ar^{\mathrm{ss}}_{\overline{K}}$ with the induced h-topology (i. e. a sieve $C$ in $\mathcal{B}$ is called covering if $\phi(C)$ is covering), then covering sieves in $\mathcal{B}$ form a Grothendieck topology on $\mathcal{B}$, the map $\mathcal{B} \to \mathcal{V}ar_{\overline{K}}$ is continuous and induces an equivalence on the corresponding toposes of sheaves. In particular, any presheaf $(U,\overline{U}) \mapsto F(U,\overline{U})$ on $\mathcal{B}$ defines an associated sheaf $aF$ on $\mathcal{V}ar_{\overline{K}}$. We will apply this to presheaves on $\mathcal{B}$ defined by certain derived de Rham complexes.

## 4. Construction of $\mathcal{A}^\natural_{\mathrm{dR}}$ and of the comparison map

We come back to the situation considered before (2.1). For any semi-stable pair $(V,\overline{V})/\mathcal{O}_{\overline{K}}$ in $\mathcal{V}ar^{\mathrm{ss}}_{\overline{K}}$, we have a morphism of log schemes $(V,\overline{V}) \to \operatorname{Spec}\mathcal{O}_K$, where $\operatorname{Spec}\mathcal{O}_K$ is endowed with the *trivial* log structure and $(V,\overline{V})$ with the canonical one, $M = \mathcal{O}_{\overline{V}} \cap j_*\mathcal{O}_V$. Consider the (pro)-complex $R\Gamma(\overline{V}, L\widehat{\Omega}^{\cdot}_{(V,\overline{V})/\mathcal{O}_K})$ (here $\overline{V}$ is endowed with the Zariski topology) (note that each $\mathrm{gr}^i_F$ has bounded, coherent cohomology, so that we would get the same pro-complex by using the étale topology). To make it functorial in $(V,\overline{V})$, we can calculate it as

$$R\Gamma(\overline{V}, L\widehat{\Omega}^{\cdot}_{(V,\overline{V})/\mathcal{O}_K}) = \Gamma(\overline{V}, \mathcal{C}(L\widehat{\Omega}^{\cdot}_{(V,\overline{V})/\mathcal{O}_K})),$$

where $\mathcal{C}$ denotes the Godement resolution associated with the set of (usual) points of $\overline{V}$ : the right hand side is the pro-object

$$\text{“}\varprojlim_i\text{”}\,\Gamma(\overline{V}, \mathcal{C}(L\Omega^{\cdot}_{(V,\overline{V})/\mathcal{O}_K}/F^i)).$$

We thus get a presheaf of pro-dga[4]

$$(V,\overline{V}) \mapsto \Gamma(\overline{V}, \mathcal{C}(L\widehat{\Omega}^{\cdot}_{(V,\overline{V})/\mathcal{O}_K}))$$

on $\mathcal{V}ar^{\mathrm{ss}}_{\overline{K}}$. Using the remark at the end of §3, we define $\mathcal{A}^\natural_{\mathrm{dR}}$ to be the associated sheaf on $\mathcal{V}ar_{\overline{K}}$ for the h-topology :

$$\mathcal{A}^\natural_{\mathrm{dR}} := a((V,\overline{V}) \mapsto \Gamma(\overline{V}, \mathcal{C}(L\widehat{\Omega}^{\cdot}_{(V,\overline{V})/\mathcal{O}_K})). \tag{4.1}$$

[4] not commutative, but having an $E_\infty$-structure

This is a sheaf of pro-$\mathcal{O}_K$-dga (even, pro-$E_\infty$-$\mathcal{O}_K$-algebras) on $\mathcal{V}ar_{\overline{K}}$ for the h-topology.

*Remark.* Instead of using Godement's resolutions, one could use Lurie's language of derived $\infty$-categories, viewing $(V,\overline{V}) \mapsto R\Gamma(\overline{V}, L\widehat{\Omega}^{\cdot}_{(V,\overline{V})/\mathcal{O}_K})$ as a presheaf of $E_\infty$-$\mathcal{O}_K$-algebras in a filtered derived $\infty$-category $\mathcal{D}$. It then makes sense to take the associated h-sheaf (with values in $\mathcal{D}$), which in turn, by Lurie's theory, defines a complex of sheaves of $\mathcal{O}_K$-modules on $\mathcal{V}ar_{\overline{K}}$, with the structure of a filtered $E_\infty$-algebra.

Let us now define the map

$$b : A_{\mathrm{dR}}\widehat{\otimes}\mathbf{Z}_p \to \mathcal{A}^{\natural}_{\mathrm{dR}}\widehat{\otimes}\mathbf{Z}_p$$

in (2.4). Actually, $b$ is short for a projective system of maps (of sheaves on $\mathcal{V}ar_{\overline{K}}$ for the h-topology) ((2.7) above)

$$b_{n,i} : (A_{\mathrm{dR}}/F^i) \otimes^L \mathbf{Z}/p^n \to (\mathcal{A}^{\natural}_{\mathrm{dR}}/F^i) \otimes^L \mathbf{Z}/p^n, \tag{4.2.1}$$

where the left hand side is viewed as a constant sheaf. The composition (of maps of log schemes)

$$(V,\overline{V}) \to \operatorname{Spec}\mathcal{O}_{\overline{K}} \to \operatorname{Spec}\mathcal{O}_K,$$

where $\operatorname{Spec}\mathcal{O}_K$ and $\operatorname{Spec}\mathcal{O}_{\overline{K}}$ are endowed with the trivial log structures[5], defines a map of (pro) (log) derived de Rham complexes on $\overline{V}$

$$L\widehat{\Omega}^{\cdot}_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}|\overline{V} \to L\widehat{\Omega}^{\cdot}_{(V,\overline{V})/\mathcal{O}_K},$$

hence, as by definition $A_{\mathrm{dR}} = L\widehat{\Omega}^{\cdot}_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ (3.8.3), a map

$$A_{\mathrm{dR}}/F^i \to \mathrm{R}\Gamma(\overline{V}, (L\widehat{\Omega}^{\cdot}_{(V,\overline{V})/\mathcal{O}_K})/F^i).$$

Sheafifying for the h-topology we get a map

$$A_{\mathrm{dR}}/F^i \to \mathcal{A}^{\natural}_{\mathrm{dR}}/F^i,$$

from which (4.2.1) is deduced by applying $\otimes^L\mathbf{Z}/p^n$. As mentioned in 2.4, we have the following crucial result :

**Theorem 2.4** (*p-adic Poincaré lemma.*) For all $i$ and $n$, $b_{n,i}$ (2.4.1) is an isomorphism in $D(\mathcal{V}ar_{\overline{K},h}, \mathcal{O}_K \otimes \mathbf{Z}/p^n\mathbf{Z})$.

---

[5]If we put on $\operatorname{Spec}\mathcal{O}_{\overline{K}}$ the canonical log structure given by the open subset $\operatorname{Spec}\overline{K}$, we would get an isomorphic transitivity triangle, see (5.1).

Finally, let us define $\mathcal{A}_{\mathrm{dR}}$ and

$$a : \mathcal{A}^{\natural}_{\mathrm{dR}} \otimes \mathbf{Q} \to A_{\mathrm{dR}}$$

in (2.4). Let $\mathcal{V}ar^{\mathrm{nc}}_{\overline{K}}$ denote the category of $\overline{K}$-pairs $(U, \overline{U})$, where $\overline{U}$ is proper and smooth over $\overline{K}$ and $U$ is a dense open subset such that the complement $\overline{U} - U$ is a divisor with normal crossings. The definition of $\mathcal{A}_{\mathrm{dR}}$ is similar to that of $\mathcal{A}^{\natural}_{\mathrm{dR}}$. We consider the presheaf

$$(U, \overline{U}) \mapsto \Gamma(\overline{U}, \mathcal{C}(L\widehat{\Omega}^{\cdot}_{(U,\overline{U})/K}))$$

on $\mathcal{V}ar^{\mathrm{nc}}_{\overline{K}}$ (where $\mathcal{C}$ denotes a Godement resolution) and define $\mathcal{A}_{\mathrm{dR}}$ to be its associated sheaf of the h-topology on $\mathcal{V}ar_{\overline{K}}$ :

$$\mathcal{A}_{\mathrm{dR}} : a((U, \overline{U}) \mapsto \Gamma(\overline{U}, \mathcal{C}(L\widehat{\Omega}^{\cdot}_{(U,\overline{U})/K}))). \tag{4.3}$$

This is a sheaf of pro-$K$-dga on $\mathcal{V}ar_{\overline{K}}$ for the h-topology, and even of pro-$\overline{K}$-dga as $L_{\overline{K}/K} = \Omega^1_{\overline{K}/K} = 0$, hence

$$L\widehat{\Omega}^{\cdot}_{(U,\overline{U})/K} \xrightarrow{\sim} L\widehat{\Omega}^{\cdot}_{(U,\overline{U}))/\overline{K}} \xrightarrow{\sim} \widehat{\Omega}^{\cdot}_{(U,\overline{U})/\overline{K}},$$

as $(U, \overline{U})$ is log smooth and integral over $\overline{K}$.

If $(V, \overline{V})$ is a semi-stable pair over $\mathcal{O}_{\overline{K}}$, $(V, \overline{V}) \otimes_{\mathcal{O}_K} K = (U, \overline{U})$ is a normal crossings pair over $\overline{K}$, and we have a canonical isomorphism (given by a canonical, functorial chain of quasi-isomorphisms)

$$L\widehat{\Omega}^{\cdot}_{(V,\overline{V})/\mathcal{O}_K} \otimes K \xrightarrow{\sim} \widehat{\Omega}^{\cdot}_{(U,\overline{U})/\overline{K}},$$

hence an isomorphism

$$\Gamma(\overline{V}, \mathcal{C}(L\widehat{\Omega}^{\cdot}_{(V,\overline{V})/\mathcal{O}_K})) \otimes \mathbf{Q} \xrightarrow{\sim} \Gamma(\overline{U}, \mathcal{C}(\widehat{\Omega}^{\cdot}_{(U,\overline{U})/\overline{K}}))$$

(given by a canonical, functorial chain of quasi-isomorphisms)[6]. As both $\mathcal{V}ar^{ss}_{\overline{K}}$ and $\mathcal{V}ar^{nc}_{\overline{K}}$ are *bases* for the h-topology of $\mathcal{V}ar_{\overline{K}}$, these isomorphisms induce an isomorphism on the associated sheaves,

$$a : \mathcal{A}^{\natural}_{\mathrm{dR}} \otimes \mathbf{Q} \xrightarrow{\sim} \mathcal{A}_{\mathrm{dR}}. \tag{4.4}$$

For $X$ in $\mathcal{V}ar_{\overline{K}}$, choose a *proper hypercovering* $V_{\cdot} \to X$, with $V_{\cdot}$ compactified into an nc simplicial pair $(V_{\cdot}, \overline{V}_{\cdot})$ over $\overline{K}$. By definition (see (c) after (2.2))

$$R\Gamma_{\mathrm{dR}}(X/\overline{K}) = R\Gamma(\overline{V}_{\cdot}, \Omega^{\cdot}_{(V_{\cdot},\overline{V}_{\cdot})/\overline{K}}),$$

[6]Here it would be more elegant to use the language of Lurie's derived $\infty$-categories alluded to in the remark above.

with corresponding pro-object

$$R\Gamma_{\mathrm{dR}}(X/\overline{K})\hat{} = R\Gamma(\overline{V}_{\cdot}, \widehat{\Omega}^{\cdot}_{(V_{\cdot},\overline{V}_{\cdot})/\overline{K}}).$$

By construction we have a canonical map

$$R\Gamma_{\mathrm{dR}}(X/\overline{K})\hat{} \to R\Gamma((V_{\cdot})_h, \mathcal{A}_{\mathrm{dR}})\hat{} \xrightarrow{\sim} R\Gamma(X_h, \mathcal{A}_{\mathrm{dR}})\hat{}. \tag{4.5}$$

Let us show that this map is an isomorphism. On $H^*$ it induces the natural map

$$H^*(\overline{V}_{\cdot}, \widehat{\Omega}^{\cdot}_{(V_{\cdot},\overline{V}_{\cdot})/\overline{K}}) \to \varinjlim H^*(\overline{Z}_{\cdot}, \widehat{\Omega}^{\cdot}_{(Z_{\cdot},\overline{Z}_{\cdot})/\overline{K}})$$

where $(Z_{\cdot}, \overline{Z}_{\cdot})$ runs through the simplicial objects of $\mathcal{V}ar^{nc}_{\overline{K}}$ endowed with an augmentation $Z_{\cdot} \to X$ which is an *h-hypercovering* (instead of a proper one). Therefore, to show that (4.5) is an isomorphisms, it suffices to show that the ind-object

$$\text{“}\varinjlim\text{”} R\Gamma(\overline{Z}_{\cdot}, \widehat{\Omega}^{\cdot}_{(Z_{\cdot},\overline{Z}_{\cdot})/\overline{K}})$$

(of the derived category of pro-dga over $\overline{K}$) is essentially constant, of value $R\Gamma(\overline{V}_{\cdot}, \widehat{\Omega}^{\cdot}_{(V_{\cdot},\overline{V}_{\cdot})/\overline{K}})$. For this, by the Lefschetz principle we may replace $\overline{K}$ by $\mathbf{C}$. Then, for $(Z_{\cdot}, \overline{Z}_{\cdot})$ mapping to $(V_{\cdot}, \overline{V}_{\cdot})$, the map

$$R\Gamma(\overline{V}_{\cdot}, \Omega^{\cdot}_{(V_{\cdot},\overline{V}_{\cdot})/\mathbf{C}}) \to R\Gamma(\overline{Z}_{\cdot}, \Omega^{\cdot}_{(Z_{\cdot},\overline{Z}_{\cdot})/\mathbf{C}})$$

by [5] underlies a morphism of mixed Hodge complexes. By cohomological descent *for the h-topology*, it induces an isomorphism on the underlying complexes of $\mathbf{C}$-vector spaces. Therefore it underlies an isomorphism of mixed Hodge complexes, and in particular is a filtered isomorphism (for the Hodge filtrations).

It's easy now to define the comparison map $\rho_{\mathrm{dR}}$ (2.1). It is the composition of several ones. In the sequel, for ease of notations we will omit the superscript $\hat{}$ indicating pro-completions.

First, we have the functoriality map

$$R\Gamma_{\mathrm{dR}}(X/K) \to R\Gamma_{\mathrm{dR}}(X_{\overline{K}}/\overline{K}). \tag{4.6}$$

Second, the isomorphism (4.5)

$$R\Gamma_{\mathrm{dR}}(X_{\overline{K}}/\overline{K}) \xrightarrow{\sim} R\Gamma(X_{\overline{K},h}, \mathcal{A}_{\mathrm{dR}}). \tag{4.7}$$

Next, we use the identification

$$R\Gamma(X_{\overline{K},h}, \mathcal{A}_{\mathrm{dR}}) \xrightarrow{\sim} R\Gamma(X_{\overline{K},h}, \mathcal{A}^{\natural}_{\mathrm{dR}}) \otimes \mathbf{Q} \tag{4.8}$$

deduced from (4.4).

Then we send the right hand side to $(R\Gamma(X_{\overline{K},h},\mathcal{A}^{\natural}_{\mathrm{dR}})\widehat{\otimes}\mathbf{Z}_p)\otimes\mathbf{Q}$ by the natural map

$$R\Gamma(X_{\overline{K},h},\mathcal{A}^{\natural}_{\mathrm{dR}})\otimes\mathbf{Q}\to(R\Gamma(X_{\overline{K},h},\mathcal{A}^{\natural}_{\mathrm{dR}})\,\widehat{\otimes}\mathbf{Z}_p)\otimes\mathbf{Q}. \tag{4.9}$$

By the Poincaré lemma (4.2) (and cohomological descent), the natural map

$$R\Gamma(X_{\overline{K}},\mathbf{Z}_p)\otimes^{L}_{\mathbf{Z}_p}((A_{\mathrm{dR}}/F^i)\widehat{\otimes}\mathbf{Z}_p)\to R\Gamma(X_{\overline{K},h},\mathcal{A}^{\natural}_{\mathrm{dR}}/F^i)\widehat{\otimes}\mathbf{Z}_p \tag{4.10}$$

is an isomorphism. Now, recall the isomorphism (2.6)

$$((A_{\mathrm{dR}}/F^i)\widehat{\otimes}\mathbf{Z}_p)\otimes\mathbf{Q}\xrightarrow{\sim}B^+_{\mathrm{dR}}/F^i.$$

Therefore, applying $\otimes\mathbf{Q}$ to (4.10) we get a filtered isomorphism

$$R\Gamma(X_{\overline{K}},\mathbf{Z}_p)\otimes B^+_{\mathrm{dR}}\xrightarrow{\sim}R\Gamma(X_{\overline{K},h},\mathcal{A}^{\natural}_{\mathrm{dR}})\widehat{\otimes}\mathbf{Q}_p, \tag{4.11}$$

where $\widehat{\otimes}\mathbf{Q}_p$ means $(\widehat{\otimes}\mathbf{Z}_p)\otimes\mathbf{Q}$.

The compositions of maps (4.6) to (4.9) is a map

$$R\Gamma_{\mathrm{dR}}(X/K)\to(R\Gamma(X_{\overline{K},h},\mathcal{A}^{\natural}_{\mathrm{dR}})\,\widehat{\otimes}\mathbf{Z}_p)\otimes\mathbf{Q}. \tag{4.12}$$

Composing (4.12) with the inverse of (4.11) (given by the Poincaré lemma), we get a map

$$R\Gamma_{\mathrm{dR}}(X_K/K)\to R\Gamma(X_{\overline{K}},\mathbf{Z}_p)\otimes B^+_{\mathrm{dR}},$$

whose associated $B^+_{\mathrm{dR}}$-linear map is the desired comparison map

$$\rho_{\mathrm{dR}}:R\Gamma_{\mathrm{dR}}(X/K)\otimes B^+_{\mathrm{dR}}\to R\Gamma(X_{\overline{K}},\mathbf{Z}_p)\otimes B^+_{\mathrm{dR}}. \tag{2.1}$$

This map is compatible with products, Galois actions, and filtrations on both sides.

The $p$-adic de Rham comparison theorem is :

**Theorem 4.13.** The map (2.1) induces an isomorphism

$$\rho_{\mathrm{dR}}:R\Gamma_{\mathrm{dR}}(X/K)\otimes B_{\mathrm{dR}}\xrightarrow{\sim}R\Gamma(X_{\overline{K}},\mathbf{Z}_p)\otimes B_{\mathrm{dR}}. \tag{2.2}$$

**5. The $p$-adic Poincaré lemma**

There are two main parts : (a) reduction to an h-local $p$-divisibility property for certain Hodge sheaves by using the Koszul filtration (b) proof of this $p$-divisibility by reduction to the case of pointed curves using de Jong and isogenies of Jacobians.

(a) *The Koszul filtration.*

Let $(U, \overline{U})$ be an ss pair over $\mathcal{O}_{\overline{K}}$ (§3 (2)). We have a transitivity triangle for log cotangent complexes

$$\mathcal{O}_{\overline{U}} \otimes L_{(\overline{K}, \mathcal{O}_{\overline{K}})/\mathcal{O}_K} \to L_{(U,\overline{U})/\mathcal{O}_K} \to L_{(U,\overline{U})/\mathcal{O}_{\overline{K}}} \to . \tag{5.1}$$

Here $L_{(\overline{K}, \mathcal{O}_{\overline{K}})/\mathcal{O}_K}$ is the log cotangent complex of the pair $(\operatorname{Spec} \overline{K}, \operatorname{Spec} \mathcal{O}_{\overline{K}})$ over $\mathcal{O}_K$ (with trivial log structure). Such a pair is not ss, but for any finite extension $K'$ of $K$, $\operatorname{Spec} \mathcal{O}_{K'}$, equipped with the log structure given by the pair $(\operatorname{Spec} K', \operatorname{Spec} \mathcal{O}_{K'})$ is a log complete intersection over $\mathcal{O}_K$, and one easily deduces from this that the natural map

$$L_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \to L_{(\overline{K}, \mathcal{O}_{\overline{K}})/\mathcal{O}_K} \tag{5.2}$$

from the non log cotangent complex to the log one is an isomorphism (if $K' \subset \overline{K}$ is a finite extension of $K$, the cone of $L_{\mathcal{O}_{K'}/\mathcal{O}_K} \to L_{(K', \mathcal{O}_{K'})/\mathcal{O}_K}$ is isomorphic to the residue field of $\mathcal{O}_{K'}$, hence the cone of (5.2) is an $\mathcal{O}_{\overline{K}}$-module which is both $p$-divisible and killed by $p$). Therefore in (5.1) we may replace $L_{(\overline{K}, \mathcal{O}_{\overline{K}})/\mathcal{O}_K}$ by $L_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$, which is isomorphic to $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$, and $L_{(U,\overline{U})/\mathcal{O}_{\overline{K}}}$ by $L_{(U,\overline{U})/(\overline{K}, \mathcal{O}_{\overline{K}})}$. Moreover, as $(U, \overline{U}) \to (\operatorname{Spec} \overline{K}, \operatorname{Spec} \mathcal{O}_{\overline{K}})$ is a filtering projective limit of log smooth integral maps, we have $L_{(U,\overline{U})/(\overline{K}, \mathcal{O}_{\overline{K}})} \xrightarrow{\sim} \Omega^1_{(U,\overline{U})/(\overline{K}, \mathcal{O}_{\overline{K}})}$, which is locally free of finite rank on $\overline{U}$. Hence (5.1) can be rewritten as a short exact sequence

$$0 \to \mathcal{O}_{\overline{U}} \otimes_{\mathcal{O}_{\overline{K}}} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \to \Omega^1_{(U,\overline{U})/\mathcal{O}_K} \to \Omega^1_{\langle U, \overline{U} \rangle} \to 0, \tag{5.3}$$

where

$$\Omega^1_{\langle U, \overline{U} \rangle} := \Omega^1_{(U,\overline{U})/(\overline{K}, \mathcal{O}_{\overline{K}})},$$

and (5.3) locally splits. Consider the (derived) Koszul filtration $(0 \subset I_0 \subset I_1 \subset \cdots \subset I_m = \operatorname{gr}^m_F L\Omega^{\cdot}_{(U,\overline{U})/\mathcal{O}_K})$ on

$$\operatorname{gr}^m_F L\Omega^{\cdot}_{(U,\overline{U})/\mathcal{O}_K} = (L\Lambda^m L_{(U,\overline{U})/\mathcal{O}_K})[-m] = (L\Lambda^m \Omega^1_{(U,\overline{U})/\mathcal{O}_K})[-m]$$

defined by (5.3), with associated graded

$$\operatorname{gr}^I_a \operatorname{gr}^m_F L\Omega^{\cdot}_{(U,\overline{U})/\mathcal{O}_K} = \operatorname{gr}^{m-a}_F A_{\mathrm{dR}}[-a] \otimes \Omega^a_{\langle U, \overline{U} \rangle}.$$

(For a short exact sequence $0 \to E' \to E \to E'' \to 0$ of flat modules over some ring, $I_a \Lambda^m E := \operatorname{Im} \Lambda^{m-a} E' \otimes \Lambda^a E \to \Lambda^m E$.)

Applying $R\Gamma(\overline{U}, -)$ and sheafifying for the h-topology, one gets a filtration $I_{\cdot}$ (with span in $[0, m]$) on $\operatorname{gr}^m_F \mathcal{A}^\natural_{\mathrm{dR}}$, with associated graded

$$\operatorname{gr}^I_a \operatorname{gr}^m_F \mathcal{A}^\natural_{\mathrm{dR}} = \operatorname{gr}^{m-a}_F A_{\mathrm{dR}}[-a] \otimes^L_{\mathcal{O}_{\overline{K}}} \mathcal{G}^a, \tag{5.4}$$

where $\mathcal{G}^a$ is the complex of h-sheaves on $\mathcal{V}ar_{\overline{K}}$ associated to $(U,\overline{U}) \mapsto R\Gamma(\overline{U},\Omega^a_{\langle U,\overline{U}\rangle})$,

$$\mathcal{G}^a := a((U,\overline{U}) \mapsto R\Gamma(\overline{U},\Omega^a_{\langle U,\overline{U}\rangle})).$$

**Lemma 5.5.**

$$(5.5.1), \qquad (\tau_{>0}\mathcal{G}^0) \otimes^L \mathbf{Z}/p = 0$$

and

$$(5.5.2) \qquad \mathcal{G}^a \otimes^L \mathbf{Z}/p = 0$$

for $a > 0$.

Note that the vanishings (5.5.1) and (5.5.2) are equivalent to saying that multiplication by $p$ on $\mathcal{H}^b\mathcal{G}^a$ is an isomorphism for all $(a,b) \neq (0,0)$, i. e. that for these values $\mathcal{H}^b\mathcal{G}^a$ is a $\mathbf{Q}$-vector space (hence a $\overline{K}$-vector space). The $p$-divisibility of $\mathcal{H}^b\mathcal{G}^0$ was first proved by Bhatt (using a method that was adapted by Beilinson to prove 5.5). (Bhatt proved that for any $f : X \to S$ proper, with $S$ affine and excellent, there exists an alteration $\pi : X' \to X$ such that, for all $i > 0$, $\pi^*H^i(X,\mathcal{O}) \subset pH^i(X',\mathcal{O})$.)

To prove that (4.2) is an isomorphism is equivalent to proving that

$$(*) \qquad \mathrm{Cone}(\mathrm{gr}^m_F A_{\mathrm{dR}} \to \mathrm{gr}^m_F \mathcal{A}^{\natural}_{\mathrm{dR}}) \otimes^L \mathbf{Z}/p = 0.$$

For a pair $(U,\overline{U})$ with $\overline{U}$ connected, $H^0(\overline{U},\mathcal{O}) = \mathcal{O}_{\overline{K}}$ (by the normality of $\overline{U}$), hence $\mathcal{H}^0\mathcal{G}^0$, which is the h-sheaf associated to $(U,\overline{U}) \mapsto H^0(\overline{U},\mathcal{O})$, is the *constant sheaf* of value $\mathcal{O}_{\overline{K}}$. Therefore,

$$\mathrm{gr}^m_F A_{\mathrm{dR}} = \mathrm{gr}^m_F A_{\mathrm{dR}} \otimes^L_{\mathcal{O}_{\overline{K}}} \mathcal{H}^0\mathcal{G}^0,$$

so that by (5.4) the left hand side of (*) has a dévissage into $(\mathrm{gr}^m_F A_{\mathrm{dR}} \otimes^L_{\mathcal{O}_{\overline{K}}} \tau_{>0}\mathcal{G}^0) \otimes^L \mathbf{Z}/p$, and $(\mathrm{gr}^{m-a}_F A_{\mathrm{dR}}[-a] \otimes^L_{\mathcal{O}_{\overline{K}}} \mathcal{G}^a) \otimes^L \mathbf{Z}/p$ for $a > 0$, which are all zero by (5.5).

(b) *Jacobians of semi-stable curves.*

The proof of (5.5) heavily uses variants of de Jong's alteration theorems, both to reduce to the following crucial lemma, and to prove it :

**Lemma 5.6.** Consider a morphism of ss-pairs over $K$ (§3, (2))

$$f : (C,\overline{C}) \to (S,\overline{S}),$$

together with a section $e : \overline{S} \to \overline{C}$ having the following properties :

(i) $f$ is a semi-stable family of curves, by which we mean that $f$ is proper and flat, its geometric fibers are semi-stable curves, and $f|S : C \to S$ is smooth, with geometrically irreducible fibers ;

(ii) The closure $D_f$ of $\overline{C}_S - C$ is an étale divisor over $S$ and $f(D_f) = \overline{S}$ (i. e. $C \to S$ is affine).

(iii) The image $e(\overline{S})$ of the section $e$ intersects the fibers of $f$ at smooth points, and $e(\overline{S}) \cap D_f = \emptyset$.

Then, after h-localization on $(S, \overline{S})$ one can find an alteration $h : (C', \overline{C}') \to (C, \overline{C})$ with $f' : (C', \overline{C}') \to (S, \overline{S})$ satisfying (i) and (ii), together with a lifting $e' : \overline{S} \to \overline{C}'$ of $e$ satisfying (iii), such that the maps

$$h^* : R^1 f_* \mathcal{O}_{\overline{C}} \to R^1 f'_* \mathcal{O}_{\overline{C}'}$$

and

$$h^* : f_* \omega_f \to f'_* \omega_{f'}$$

are divisible by $p$, where $\omega := \Omega^1_{(-,-)/(S,\overline{S})}$.

*Remark* 5.6.1. By the local structure of nodal curves over regular pairs (cf. ([3],2.23), ([15], 1.9)) we have

$$\omega_f = f^! \mathcal{O}_{\overline{S}}[-1] \otimes \mathcal{O}_{\overline{C}}(D_f).$$

Notice that $f_* \mathcal{O}_{\overline{C}}(-D_f) = 0$ since $f(D_f) = \overline{S}$, hence $Rf_* \mathcal{O}_{\overline{C}}(-D_f) = R^1 f_* \mathcal{O}_{\overline{C}}(-D_f)[-1]$, with $R^1 f_* \mathcal{O}_{\overline{C}}(-D_f)$ locally free of finite type, and dually $R^1 f_* \omega_f = 0$, with $f_* \omega_f$ dual to $R^1 f_* \mathcal{O}_{\overline{C}}(-D_f)$.

*Proof of 5.6.* Let us first sketch the construction of $(C', \overline{C}')$. Consider the exact sequence of group schemes on $\overline{S}$,

$$0 \to T \to J^\flat \to J \to 0,$$

where

$$J = Pic^0_{\overline{C}/\overline{S}}$$

is the Picard scheme parametrizing line bundles on $\overline{C}$ whose restriction to the normalization of each irreducible component of any geometric fiber of $f$ is of degree zero,

$$J^\flat = Pic^0_{(\overline{C}, D_f)/\overline{S}}$$

the Picard scheme parametrizing line bundles as above together with a trivialization along $D_f$, and $T$ the torus

$$T = (f_* \mathbf{G}_{m,D_f})/\mathbf{G}_m.$$

The group schemes $J$ and $J^\flat$ are semiabelian schemes. The section $e : \overline{S} \to \overline{C}$ defines an Abel-Jacobi map

$$i : C \to J^\flat_S,$$

$x \mapsto \mathcal{O}_{\overline{C}}(x - e)$ (an immersion as soon as the genus of $\overline{C}_S$ is at least 1). We first define $\tilde{C}$ to be the normalized $i$-pull-back of $p : J^\flat_S \to J^\flat_S$ :

$$\begin{tikzcd} \tilde{C} \arrow[r] \arrow[d] & J^\flat_S \arrow[d, "p"] \\ C \arrow[r, "i"] & J^\flat_S \end{tikzcd} .$$

Then one defines $\overline{\tilde{C}}$ to be the normalization of $\overline{C}$ in $\tilde{C}$ :

$$\begin{tikzcd} \overline{\tilde{C}} \arrow[d] & \tilde{C} \arrow[l] \arrow[d] \\ \overline{C} & C \arrow[l] \end{tikzcd} .$$

Finally, using a refinement of de Jong's theorem due to Temkin [19] (or doing it directly as does Beilinson), one shows that working h-locally on $\overline{S}$ one can find an alteration $h : (C', \overline{C}') \to (\tilde{C}, \overline{\tilde{C}})$ with a lifting $e'$ of $e$ such that $f' : (C', \overline{C}') \to (S, \overline{S})$ satisfies the conditions (i) - (iii). This $h$ has the desired $p$-divisibility property for $h^*$. This is proved by playing a clever game with $J$, $J^\flat$, $J'$, $J'^\flat$. There are two steps :

(a) The commutative square

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "h"] & J^\flat_S \arrow[d, "p"] \\ C \arrow[r] & J^\flat_S \end{tikzcd}$$

induces a commutative square

$$\begin{tikzcd} J'^\flat_S \arrow[r] \arrow[d, "h_*"] & J^\flat_S \arrow[d, "p"] \\ J^\flat_S \arrow[r, "\mathrm{Id}"] & J^\flat_S \end{tikzcd} ,$$

hence $h_* : J'^\flat_S \to J^\flat_S$ is divisible by $p$. Therefore, as $\overline{S}$ is normal, by Faltings-Chai [9],

$$h_* : J'^\flat \to J^\flat$$

is divisible by $p$. By 5.6.1 the induced map on the Lie algebras is the dual of $h^* : f_*\omega_f \to f'_*\omega_{f'}$, which is therefore divisible by $p$.

(b) Similarly, the commutative square

$$\begin{tikzcd} \overline{C}'_S \arrow[r] \arrow[d, "h"] & J_S \arrow[d, "p"] \\ \overline{C}_S \arrow[r] & J_S \end{tikzcd}$$

induces a commutative square

$$\begin{tikzcd} J'_S \arrow[r] \arrow[d, "h_*"] & J_S \arrow[d, "p"] \\ J_S \arrow[r, "\mathrm{Id}"] & J_S \end{tikzcd} ,$$

so $h_* : J'_S \to J_S$ is divisible by $p$. By the self-duality of Jacobians, $h^* : J_S \to J'_S$ is identified with the dual of $h_*$, hence also divisible by $p$. By Faltings-Chai the same is true of $h^* : J \to J'$. But the map induced by $h^*$ on the Lie algebras is $h^* : R^1f_*\mathcal{O}_C \to R^1f'_*\mathcal{O}_{C'}$, which is thus divisible by $p$.

*End of proof of 5.5.* It's enough to show that any ss-pair $(U, \overline{U})$ *over* $K$ admits an h-covering $(U', \overline{U}') \to (U, \overline{U})$ by an ss-pair that kills $\tau_{>0}R\Gamma(\overline{U}, \mathcal{O}_{\overline{U}}) \otimes^L \mathbf{Z}/p$ and $R\Gamma(\overline{U}, \Omega^a_{(U,\overline{U})/\mathcal{O}_{K_U}}) \otimes^L \mathbf{Z}/p$ for $a > 0$.

For this one proceeds by induction on the dimension of $U$, using 5.6 and the following de Jong type *preparation lemma* :

**Lemma 5.7**. Let $(U, \overline{U})$ be an ss-pair over $K$. Then there exists a diagram of ss-pairs over $\mathcal{O}_K$ :

(5.7.1)
$$\begin{tikzcd} (C, \overline{C}) \arrow[r, "h"] \arrow[d, "f"] & (U, \overline{U}) \\ (S, \overline{S}) & \end{tikzcd}$$

where $h$ is an h-covering with dim $U$ = dim $C$, and $f$ is a morphism of ss-pairs admitting a section $e$ satisfying conditions (i) - (iii) of 5.6.

Thus, thanks to 5.6 and 5.7 we may assume that we have a diagram (5.7.1) and an alteration $h : (C', \overline{C}') \to (C, \overline{C})$ satisfying the properties of 5.6. To finish the proof of 5.5, it suffices to show that there exists an h-covering $(S', \overline{S}') \to (S, \overline{S})$ such that the composition

$$(C', \overline{C'})_{(S', \overline{S}')} \to (C', \overline{C'}) \to (C, \overline{C})$$

kills $\tau_{>0}R\Gamma(-,\mathcal{O}_{\overline{C}})\otimes^L \mathbf{Z}/p$ and $R\Gamma(-,\Omega^a_{(C,\overline{C})})\otimes^L \mathbf{Z}/p$ for $a>0$, where differentials are log differentials relative to $(K_C,\mathcal{O}_{K_C})$, etc. Again, we use the Koszul filtration on $\Omega^a_{(C,\overline{C})}$ relative to $f$, which has only one step : $I_0 = f^*\Omega^a_{(S,\overline{S})}$ and $I_1 = \Omega^a_{(C,\overline{C})}$,

$$0 \to f^*\Omega^a_{(S,\overline{S})} \to \Omega^a_{(C,\overline{C})} \to f^*\Omega^{a-1}_{(S,\overline{S})} \otimes \omega_f \to 0.$$

Applying $Rf_*$ and using that $R^1f_*\omega_f = 0$ (5.6.1), we get a triangle

$$\Omega^a_{(S,\overline{S})} \otimes Rf_*\mathcal{O}_{\overline{C}} \to Rf_*\Omega^a_{(C,\overline{C})} \to \Omega^{a-1}_{(S,\overline{S})} \otimes f_*\omega_f \to .$$

The section $e$ of $f$ splits off $Rf_*\Omega^a_{(C,\overline{C})}$ into

$$Rf_*\Omega^a_{(C,\overline{C})} = \Omega^a_{(S,\overline{S})} \oplus L,$$

and the above triangle sits in a 9-diagram

$$\begin{tikzcd}
\Omega^a_{(S,\overline{S})} \otimes R^1f_*\mathcal{O}_{\overline{C}}[-1] \arrow[r] & L \arrow[r] & \Omega^{a-1}_{(S,\overline{S})} \otimes f_*\omega_f \to . \\
\Omega^a_{(S,\overline{S})} \otimes Rf_*\mathcal{O}_{\overline{C}} \arrow[r] \arrow[u] & Rf_*\Omega^a_{(C,\overline{C})} \arrow[r] \arrow[u] & \Omega^{a-1}_{(S,\overline{S})} \otimes f_*\omega_f \to \arrow[u, "Id"'] \\
\Omega^a_{(S,\overline{S})} \arrow[r, "Id"] \arrow[u] & \Omega^a_{(S,\overline{S})} \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Thus $L$ is isomorphic to the cone of a coboundary map :

$$L \simeq \mathrm{Cone}(\partial : \Omega^{a-1}_{(S,\overline{S})} \otimes f_*\omega_f \to \Omega^a_{(S,\overline{S})} \otimes R^1f_*\mathcal{O}_{\overline{C}})[-1]$$

(*a Kodaira-Spencer complex*). The map $h^* : Rf_*\Omega^a_{(C,\overline{C})} \to Rf_*\Omega^a_{(C',\overline{C}')}$ respects the above decompositions and one deduces from 5.6 that $h^* \otimes^L \mathbf{Z}/p : L\otimes^L\mathbf{Z}/p \to L'\otimes^L\mathbf{Z}/p$ is zero, hence $h^* : R\Gamma(\overline{S},L)\otimes^L\mathbf{Z}/p \to R\Gamma(\overline{S},L')\otimes^L\mathbf{Z}/p$ is also zero. Induction disposes of the other summand $R\Gamma(\overline{S},\Omega^a_{(S,\overline{S})})$.

## 6. Sketch of proof of the comparison isomorphism theorem

The crucial verification is for $\mathbf{G}_m$ and $H^1$. It is made by an explicit Čech cocycle calculation.

**Lemma 6.1.** For $X = \mathbf{G}_{mK} = \operatorname{Spec} K[t,t^{-1}]$ the comparison map

$$\rho : H^1_{\mathrm{dR}}(X/K) \otimes B_{\mathrm{dR}} \to H^1(X_{\overline{K}},\mathbf{Z}_p) \otimes B_{\mathrm{dR}}$$

is a filtered isomorphism, sending the class of $dt/t$ in $F^1H^1_{\mathrm{dR}}$ to $\kappa\otimes\varpi$, where $\kappa\in H^1(X_{\overline{K}},\mathbf{Z}_p(1))$ is the class of the Kummer $\mathbf{Z}_p(1)$-torsor $\varprojlim(t^{1/p^n})$ over $X_{\overline{K}}$, and $\varpi=\varepsilon^{-1}\otimes\log[\varepsilon]\in\mathfrak{m}_{\mathrm{dR}}(-1)$ is the canonical Fontaine element, which maps each generator $\epsilon=(\epsilon_n)$ of $\mathbf{Z}_p(1)$ to $\log([\varepsilon])\in\mathfrak{m}_{\mathrm{dR}}$ (a $p$-adic analogue of $2\pi i$).

Once 6.1 is established, one deduces compatibility of $\rho$ with Gysin maps for codimension one closed immersions of smooth varieties, hence for classes of hyperplane sections. As usual, by Poincaré duality this yields the case of projective smooth varieties. Then one deduces the result for the complement of a strict normal crossings divisor in a projective smooth scheme, and finally the general case by Hironaka and cohomological descent.

## 7. Quick review of derived log cotangent and de Rham complexes

Let $\mathcal{T}$ be a topos. A *pre-log ring in* $\mathcal{T}$ (or *pre-log structure on* $\mathcal{T}$) is a multiplicative homomorphism $\alpha:L\to A$, where $L$ a commutative monoid with unit, and $A$ a ring. Homomorphisms of such objects are defined in the obvious way. A pre-log structure $\alpha:L\to A$ is called a *log structure* if $\alpha$ induces an isomorphism $\alpha^{-1}(A^*)\xrightarrow{\sim}A^*$, where $A^*$ is the group of units in $A$. The forgetful functor from log rings to pre-log rings admits a left adjoint, denoted $(L\to A)\mapsto(L\to A)^a=(L^a\to A)$. There is a canonical adjunction map $L\to L^a$, $(L\to A)^a$ is called the *associated log ring* (or *log structure*).

For a map of pre-log rings $(L\to A)\to(M\to B)$ one defines its $B$-module of Kähler differentials

$$\Omega^1_{(M\to B)/(L\to A)}:=(\Omega^1_{B/A}\oplus(B\otimes_{\mathbf{Z}}(\mathrm{Coker}(L^{\mathrm{gp}}\to M^{\mathrm{gp}}))))/R,$$

where the exponent gp means the group envelope, and $R$ is the subgroup generated by the image of $m\mapsto(d\alpha(m),0)-(0,\alpha(m)\otimes m)$, with an $A$-derivation

$$d:B\to\Omega^1_{(M\to B)/(L\to A)},b\mapsto\text{image of }d_{B/A}(b),$$

and a homomorphism

$$d\log:M\to\Omega^1_{(M\to B)/(L\to A)},m\mapsto\text{image of }(0,1\otimes m)$$

satisfying

$$\alpha(m)d\log m=d\alpha(m)$$

universal among similar pairs $(D:B\to E,D\log:M\to E)$ called $(L\to A)$-*log derivations* from $(M\to B)$ to a $B$-module $E$. We have :

$$\Omega^1_{(M\to B)/(L\to A)}\xrightarrow{\sim}\Omega^1_{(M\to B)^a/(L\to A)^a}.$$

For a map of log schemes $(X, M) \to (Y, L)$[7],

$$\Omega^1_{(X,M)/(Y,L)} := \Omega^1_{(M\to\mathcal{O}_X)/(f^{-1}L\to f^{-1}\mathcal{O}_Y)}.$$

The following construction is due to Gabber ([18], §8). Fix a prelog ring $(L \to A)$ in $\mathcal{T}$. The forgetful functor $U$ from the category of $(L \to A)$-prelog rings to the category of pairs of sheaves of sets admits a left adjoint

$$(X, Y) \mapsto T_{(L\to A)}(X, Y) := (L \oplus \mathbf{N}^{(X)} \to A[X \coprod Y]),$$

abridged to $T(X, Y)) =$ cobase change of $(\mathbf{N}^{(X)} \to \mathbf{Z}[X \coprod Y])$ by $(0 \to \mathbf{Z}) \to (L \to A)$ :

$$\begin{tikzcd}
(\mathbf{N}^{(X)} \to \mathbf{Z}[X \coprod Y]) \arrow[r] & (L \oplus \mathbf{N}^{(X)} \to A[X \coprod Y]) . \\
(0 \to \mathbf{Z}) \arrow[u] \arrow[r] & (L \to A) \arrow[u]
\end{tikzcd}$$

$T(X, Y)$ is called the *free* $(L \to A)$*-log algebra generated by* $(X, Y)$. Note that :

$$\Omega^1_{T(X,Y)/(L\to A)} = B^{(X)} \oplus B^{(Y)}$$

(basis made from $dy$, $y \in Y$, and $d\log x$, $x \in X$).

This pair of adjoint functors gives rise to a standard resolution, called the *canonical free resolution*

$$P_{(L\to A)}(M \to B) = (T, U).(M \to B),$$

where $(T, U)_. : [n] \mapsto (TU)^{[n]}$ is the canonical simplicial object defined by the pair $(T, U)$ ([13], I 1.5.2). Each component $P_n$ is free on the underlying pair of $P_{n-1}$, and *resolution* means that the pair of underlying simplicial (sheaves of) sets of $P$ is augmented to $(M, B)$ by a pair of quasi-isomorphisms.

The *log cotangent complex of* $(M \to B)$ *over* $(L \to A)$ is defined by

$$L_{(M\to B)/(L\to A)} := \Omega^1_{P_{(L\to A)}(M\to B)/(L\to A)} \otimes B$$

(tensor product taken over the underlying (simplicial) ring of $P_{(L\to A)}(M \to B)$). For a map $f : (X, M) \to (Y, L)$ of log schemes, the *log cotangent complex of* $(X, M)$ *over* $(Y, N)$ is defined by

$$L_{(X,M)/(Y,L)} := L_{(M\to\mathcal{O}_X)/(f^{-1}L\to f^{-1}\mathcal{O}_Y)}.$$

[7]see §3 (b) *Logarithmic variants* for the notation.

An important (nontrivial) point is that passing to the associated log rings does not change the log cotangent complex : the canonical map

$$L_{(M\to B)/(L\to A)} \xrightarrow{\sim} L_{(M\to B)^a/(L\to A)^a}$$

is an isomorphism ([18], 8.20).

Another useful fact is that for $f : (X, M) \to (Y, L)$ log smooth,

$$\mathrm{Cone}\,(L_{(X,M)/(Y,L)} \to \Omega^1_{(X,M)/(Y,L)})$$

is cohomologically concentrated in degrees $\leq -3$, zero if $f$ is integral ([18], 8.32, 8.34).

The *derived log de Rham complex* of $(M \to B)$ over $(L \to A)$ is defined by

$$L\Omega^{\cdot}_{(M\to B)/(L\to A)} := \mathbf{s}\Omega^{\cdot}_{P_{(L\to A)}(M\to B)/(L\to A)}$$

and the pro-completed one by the projective system

$$L\widehat{\Omega}^{\cdot}_{(M\to B)/(L\to A)} := \text{“}\varprojlim_i\text{”}\, L\Omega^{\cdot}_{(M\to B)/(L\to A)}/F^i,$$

where

$$F^i = \mathbf{s}\Omega^{\geq i}_{P/(L\to A)}.$$

For a map of log schemes $(X, M) \to (Y, L)$, the derived log de Rham complex is defined by

$$L\Omega^{\cdot}_{(X,M)/(Y,L)} := L\Omega^{\cdot}_{(M\to\mathcal{O}_X)/(f^{-1}(L)\to f^{-1}(\mathcal{O}_Y))}.$$

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